

**Sixth Term Examination Papers**

**9475**

**MATHEMATICS 3**

Afternoon

**THURSDAY 23 JUNE 2016**

Time: 3 hours

\* 4 1 5 6 0 2 8 5 6 6 \*

Additional Materials: Answer Booklet  
Formulae Booklet

### **INSTRUCTIONS TO CANDIDATES**

**Please read this page carefully, but do not open this question paper until you are told that you may do so.**

Write your name, centre number and candidate number in the spaces on the answer booklet.

Begin each answer on a new page.

Write the numbers of the questions you answer in the order attempted on the front of the answer booklet.

### **INFORMATION FOR CANDIDATES**

Each question is marked out of 20. There is no restriction of choice.

All questions attempted will be marked.

Your final mark will be based on the **six** questions for which you gain the highest marks.

You are advised to concentrate on no more than **six** questions. Little credit will be given for fragmentary answers.

You are provided with a Mathematical Formulae Booklet.

**Calculators are not permitted.**

**Please wait to be told you may begin before turning this page.**

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This question paper consists of 8 printed pages and 4 blank pages.

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## Section A: Pure Mathematics

1 Let

$$I_n = \int_{-\infty}^{\infty} \frac{1}{(x^2 + 2ax + b)^n} dx,$$

where  $a$  and  $b$  are constants with  $b > a^2$ , and  $n$  is a positive integer.

(i) By using the substitution  $x + a = \sqrt{b - a^2} \tan u$ , or otherwise, show that

$$I_1 = \frac{\pi}{\sqrt{b - a^2}}.$$

(ii) Show that  $2n(b - a^2) I_{n+1} = (2n - 1) I_n$ .

(iii) Hence prove by induction that

$$I_n = \frac{\pi}{2^{2n-2}(b - a^2)^{n-\frac{1}{2}}} \binom{2n-2}{n-1}.$$

2 The distinct points  $P(ap^2, 2ap)$ ,  $Q(aq^2, 2aq)$  and  $R(ar^2, 2ar)$  lie on the parabola  $y^2 = 4ax$ , where  $a > 0$ . The points are such that the normal to the parabola at  $Q$  and the normal to the parabola at  $R$  both pass through  $P$ .

(i) Show that  $q^2 + qp + 2 = 0$ .

(ii) Show that  $QR$  passes through a certain point that is independent of the choice of  $P$ .

(iii) Let  $T$  be the point of intersection of  $OP$  and  $QR$ , where  $O$  is the coordinate origin. Show that  $T$  lies on a line that is independent of the choice of  $P$ .

Show further that the distance from the  $x$ -axis to  $T$  is less than  $\frac{a}{\sqrt{2}}$ .

- 3 (i) Given that

$$\int \frac{x^3 - 2}{(x + 1)^2} e^x dx = \frac{P(x)}{Q(x)} e^x + \text{constant},$$

where  $P(x)$  and  $Q(x)$  are polynomials, show that  $Q(x)$  has a factor of  $x + 1$ .

Show also that the degree of  $P(x)$  is exactly one more than the degree of  $Q(x)$ , and find  $P(x)$  in the case  $Q(x) = x + 1$ .

- (ii) Show that there are no polynomials  $P(x)$  and  $Q(x)$  such that

$$\int \frac{1}{x + 1} e^x dx = \frac{P(x)}{Q(x)} e^x + \text{constant}.$$

You need consider only the case when  $P(x)$  and  $Q(x)$  have no common factors.

- 4 (i) By considering  $\frac{1}{1 + x^r} - \frac{1}{1 + x^{r+1}}$  for  $|x| \neq 1$ , simplify

$$\sum_{r=1}^N \frac{x^r}{(1 + x^r)(1 + x^{r+1})}.$$

Show that, for  $|x| < 1$ ,

$$\sum_{r=1}^{\infty} \frac{x^r}{(1 + x^r)(1 + x^{r+1})} = \frac{x}{1 - x^2}.$$

- (ii) Deduce that

$$\sum_{r=1}^{\infty} \operatorname{sech}(ry) \operatorname{sech}((r + 1)y) = 2e^{-y} \operatorname{cosech}(2y)$$

for  $y > 0$ .

Hence simplify

$$\sum_{r=-\infty}^{\infty} \operatorname{sech}(ry) \operatorname{sech}((r + 1)y),$$

for  $y > 0$ .

- 5 (i) By considering the binomial expansion of  $(1+x)^{2m+1}$ , prove that

$$\binom{2m+1}{m} < 2^{2m},$$

for any positive integer  $m$ .

- (ii) For any positive integers  $r$  and  $s$  with  $r < s$ ,  $P_{r,s}$  is defined as follows:  $P_{r,s}$  is the product of all the prime numbers greater than  $r$  and less than or equal to  $s$ , if there are any such primes numbers; if there are no such primes numbers, then  $P_{r,s} = 1$ .

For example,  $P_{3,7} = 35$ ,  $P_{7,10} = 1$  and  $P_{14,18} = 17$ .

Show that, for any positive integer  $m$ ,  $P_{m+1,2m+1}$  divides  $\binom{2m+1}{m}$ , and deduce that

$$P_{m+1,2m+1} < 2^{2m}.$$

- (iii) Show that, if  $P_{1,k} < 4^k$  for  $k = 2, 3, \dots, 2m$ , then  $P_{1,2m+1} < 4^{2m+1}$ .

- (iv) Prove that  $P_{1,n} < 4^n$  for  $n \geq 2$ .

- 6 Show, by finding  $R$  and  $\gamma$ , that  $A \sinh x + B \cosh x$  can be written in the form  $R \cosh(x + \gamma)$  if  $B > A > 0$ . Determine the corresponding forms in the other cases that arise, for  $A > 0$ , according to the value of  $B$ .

Two curves have equations  $y = \operatorname{sech} x$  and  $y = a \tanh x + b$ , where  $a > 0$ .

- (i) In the case  $b > a$ , show that if the curves intersect then the  $x$ -coordinates of the points of intersection can be written in the form

$$\pm \operatorname{arcosh} \left( \frac{1}{\sqrt{b^2 - a^2}} \right) - \operatorname{artanh} \frac{a}{b}.$$

- (ii) Find the corresponding result in the case  $a > b > 0$ .

- (iii) Find necessary and sufficient conditions on  $a$  and  $b$  for the curves to intersect at two distinct points.

- (iv) Find necessary and sufficient conditions on  $a$  and  $b$  for the curves to touch and, given that they touch, express the  $y$ -coordinate of the point of contact in terms of  $a$ .

- 7 Let  $\omega = e^{2\pi i/n}$ , where  $n$  is a positive integer. Show that, for any complex number  $z$ ,

$$(z - 1)(z - \omega) \cdots (z - \omega^{n-1}) = z^n - 1.$$

The points  $X_0, X_1, \dots, X_{n-1}$  lie on a circle with centre  $O$  and radius 1, and are the vertices of a regular polygon.

- (i) The point  $P$  is equidistant from  $X_0$  and  $X_1$ . Show that, if  $n$  is even,

$$|PX_0| \times |PX_1| \times \cdots \times |PX_{n-1}| = |OP|^n + 1,$$

where  $|PX_k|$  denotes the distance from  $P$  to  $X_k$ .

Give the corresponding result when  $n$  is odd. (There are two cases to consider.)

- (ii) Show that

$$|X_0X_1| \times |X_0X_2| \times \cdots \times |X_0X_{n-1}| = n.$$

- 8 (i) The function  $f$  satisfies, for all  $x$ , the equation

$$f(x) + (1 - x)f(-x) = x^2.$$

Show that  $f(-x) + (1 + x)f(x) = x^2$ . Hence find  $f(x)$  in terms of  $x$ . You should verify that your function satisfies the original equation.

- (ii) The function  $K$  is defined, for  $x \neq 1$ , by

$$K(x) = \frac{x + 1}{x - 1}.$$

Show that, for  $x \neq 1$ ,  $K(K(x)) = x$ .

The function  $g$  satisfies the equation

$$g(x) + xg\left(\frac{x + 1}{x - 1}\right) = x \quad (x \neq 1).$$

Show that, for  $x \neq 1$ ,  $g(x) = \frac{2x}{x^2 + 1}$ .

- (iii) Find  $h(x)$ , for  $x \neq 0$ ,  $x \neq 1$ , given that

$$h(x) + h\left(\frac{1}{1 - x}\right) = 1 - x - \frac{1}{1 - x} \quad (x \neq 0, \quad x \neq 1).$$

## Section B: Mechanics

- 9 Three pegs  $P$ ,  $Q$  and  $R$  are fixed on a smooth horizontal table in such a way that they form the vertices of an equilateral triangle of side  $2a$ . A particle  $X$  of mass  $m$  lies on the table. It is attached to the pegs by three springs,  $PX$ ,  $QX$  and  $RX$ , each of modulus of elasticity  $\lambda$  and natural length  $l$ , where  $l < \frac{2}{\sqrt{3}}a$ . Initially the particle is in equilibrium. Show that the extension in each spring is  $\frac{2}{\sqrt{3}}a - l$ .

The particle is then pulled a small distance directly towards  $P$  and released. Show that the tension  $T$  in the spring  $RX$  is given by

$$T = \frac{\lambda}{l} \left( \sqrt{\frac{4a^2}{3} + \frac{2ax}{\sqrt{3}} + x^2} - l \right),$$

where  $x$  is the displacement of  $X$  from its equilibrium position.

Show further that the particle performs approximate simple harmonic motion with period

$$2\pi \sqrt{\frac{4mla}{3(4a - \sqrt{3}l)\lambda}}.$$

- 10 A smooth plane is inclined at an angle  $\alpha$  to the horizontal. A particle  $P$  of mass  $m$  is attached to a fixed point  $A$  above the plane by a light inextensible string of length  $a$ . The particle rests in equilibrium on the plane, and the string makes an angle  $\beta$  with the plane.

The particle is given a horizontal impulse parallel to the plane so that it has an initial speed of  $u$ . Show that the particle will not immediately leave the plane if  $ag \cos(\alpha + \beta) > u^2 \tan \beta$ .

Show further that a necessary condition for the particle to perform a complete circle whilst in contact with the plane is  $6 \tan \alpha \tan \beta < 1$ .

**11** A car of mass  $m$  travels along a straight horizontal road with its engine working at a constant rate  $P$ . The resistance to its motion is such that the acceleration of the car is zero when it is moving with speed  $4U$ .

- (i) Given that the resistance is proportional to the car's speed, show that the distance  $X_1$  travelled by the car while it accelerates from speed  $U$  to speed  $2U$ , is given by

$$\lambda X_1 = 2 \ln \frac{9}{5} - 1,$$

where  $\lambda = P/(16mU^3)$ .

- (ii) Given instead that the resistance is proportional to the square of the car's speed, show that the distance  $X_2$  travelled by the car while it accelerates from speed  $U$  to speed  $2U$  is given by

$$\lambda X_2 = \frac{4}{3} \ln \frac{9}{8}.$$

- (iii) Given that  $3.17 < \ln 24 < 3.18$  and  $1.60 < \ln 5 < 1.61$ , determine which is the larger of  $X_1$  and  $X_2$ .



## Section C: Probability and Statistics

- 12** Let  $X$  be a random variable with mean  $\mu$  and standard deviation  $\sigma$ . *Chebyshev's inequality*, which you may use without proof, is

$$P(|X - \mu| > k\sigma) \leq \frac{1}{k^2},$$

where  $k$  is any positive number.

- (i) The probability of a biased coin landing heads up is 0.2. It is thrown  $100n$  times, where  $n$  is an integer greater than 1. Let  $\alpha$  be the probability that the coin lands heads up  $N$  times, where  $16n \leq N \leq 24n$ .

Use Chebyshev's inequality to show that

$$\alpha \geq 1 - \frac{1}{n}.$$

- (ii) Use Chebyshev's inequality to show that

$$1 + n + \frac{n^2}{2!} + \cdots + \frac{n^{2n}}{(2n)!} \geq \left(1 - \frac{1}{n}\right) e^n.$$

- 13** Given a random variable  $X$  with mean  $\mu$  and standard deviation  $\sigma$ , we define the *kurtosis*,  $\kappa$ , of  $X$  by

$$\kappa = \frac{E((X - \mu)^4)}{\sigma^4} - 3.$$

Show that the random variable  $X - a$ , where  $a$  is a constant, has the same kurtosis as  $X$ .

- (i) Show by integration that a random variable which is Normally distributed with mean 0 has kurtosis 0.
- (ii) Let  $Y_1, Y_2, \dots, Y_n$  be  $n$  independent, identically distributed, random variables with mean 0, and let  $T = \sum_{r=1}^n Y_r$ . Show that

$$E(T^4) = \sum_{r=1}^n E(Y_r^4) + 6 \sum_{r=1}^{n-1} \sum_{s=r+1}^n E(Y_s^2)E(Y_r^2).$$

- (iii) Let  $X_1, X_2, \dots, X_n$  be  $n$  independent, identically distributed, random variables each with kurtosis  $\kappa$ . Show that the kurtosis of their sum is  $\frac{\kappa}{n}$ .

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